

BOUNDARY-LAYER AND INTERIOR SEPARATIONS IN THE TAYLOR–COUETTE–POISEUILLE FLOW

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ABSTRACT. In this article, we derive a rigorous characterization of the boundary-layer and interior separations in the Taylor-Couette-Poiseuille (TCP) flow. The results obtained provide a rigorous characterization on how, when and where the propagating Taylor vortices (PTV) are generated. In particular, contrary to what is commonly believed, we show that the PTV do not appear after the first dynamical bifurcation, and they appear only when the Taylor number is further increased to cross another critical value so that a structural bifurcation occurs. This structural bifurcation corresponds to the boundary-layer and interior separations of the flow structure in the physical space.

1. INTRODUCTION

Consider a viscous fluid between two coaxial rotating cylinders. The base (Couette) flow becomes unstable as soon as the rotation speed of the inner cylinder exceeds a critical value. This instability gives rise to a stationary axisymmetric counter-rotating vortices that fill the whole annular region. The associated flow is referred to as Taylor–Couette (TC) flow. When a through flow driven by a pressure gradient along the rotating axis is added, the resulting system can exhibit both convective and absolute instabilities. The base flow consists of a superposition of circular Couette flow and annular Poiseuille flow, called Couette-Poiseuille (CP) flow. The axial through-flow suppresses the basic stationary instability, and as the axial pressure gradient increases while the rotation speed of the inner cylinder is held fixed, the first bifurcation gives rise to a traveling train of axisymmetric Taylor vortices, commonly referred to as propagating Taylor vortices (PTV). Henceforth, the term Taylor–Couette–Poiseuille (TCP) flow is used to refer to all hydrodynamic phenomena pertaining to the open system described above; see among others Raguin and Georgiadis [13] and the references therein.

In this article we rigorously characterize the stability and transitions of the CP and TCP flows in both the physical and phase spaces. The main focus is on a rigorous characterization on how, when and where the PTV are generated.

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We first examine the existence, explicit formula, and basin of attractions of the secondary flows. The existence of a bifurcation to a steady state solution is classical, and can be proved by the classical Krasnoselskii theorem. The new ingredients here are the explicit formula and the basin of attraction of the bifurcated solutions. These enable us to determine, in the next two sections, the asymptotic structure in the physical space of the solutions of the problem, leading to justification of the flow structure of the TCP flow.

The main result of this article is on the rigorous characterization of the boundary-layer and interior separations, associated with the TCP flows. In particular, contrary to what is commonly believed, we show that the PTV do not appear after the first dynamical bifurcation, and they appear only when the Taylor number is further increased to cross another critical value so that a structural bifurcation occurs. This structural bifurcation corresponds to the boundary layer and interior separations of the flow structure in the physical space. This study gives another example linking the dynamics of fluid flows to the structure and its transitions in the physical spaces.

The analysis is based on a geometric theory of two-dimensional (2D) incompressible flows, and a bifurcation theory for nonlinear partial differential equations and both developed recently by the authors; see respectively [11] and [12] and the references therein.

The geometric theory of 2D incompressible flows was initiated by the authors to study the structure and its stability and transitions of 2-D incompressible fluid flows in the physical spaces. This program of study consists of research in two directions: 1) the study of the structure and its transitions/evolutions of divergence-free vector fields, and 2) the study of the structure and its transitions of the flow fields of fluid flows governed by the Navier-Stokes equations or the Euler equations. The study in the first direction is more kinematic in nature, and the results and methods developed can naturally be applied to other problems of mathematical physics involving divergence-free vector fields. In fluid dynamics context, the study in the second direction involves specific connections between the solutions of the Navier-Stokes or the Euler equations and flow structure in the physical space. In other words, this area of research links the kinematics to the dynamics of fluid flows. This is unquestionably an important and difficult problem. Progresses have been made in several directions. First, a new rigorous characterization of boundary-layer separations for 2D viscous incompressible flows is developed recently by the authors, in collaboration in part with Michael Ghil; see [12] and the references therein. Another example in this area is the structure (e.g. rolls) in the physical space in the Rayleigh-Bénard convection, using the structural stability theorem developed in Area 1) together with application of the aforementioned bifurcation theory; see [9, 12]. We would like to mention that this article gives another example in this program of research.

The dynamic bifurcation theory is centered at a new notion of bifurcation, called attractor bifurcation for dynamical systems, both finite dimensional

and infinite dimensional, together with new new strategies for the Lyapunov-Schmidt reduction and the center manifold reduction procedures. The bifurcation theory has been applied to various problems from science and engineering, including, in particular, the Kuramoto-Sivashinsky equation, the Cahn-Hilliard equation, the Ginzburg-Landau equation, Reaction-Diffusion equations in Biology and Chemistry, the Bénard convection problem and the Taylor problem in classical fluid dynamics, and the some geophysical fluid dynamics problems. We mention the interested readers to a recent monograph by the authors [11] and the references therein.

The paper is organized as follows. Section 2 introduces the physical problem, TCP problem. Section 3 proves an abstract dynamical bifurcation theorem, which will be used in Section 4 to derive an explicit form and the basin of attractions of the bifurcated solutions. Sections 5 and 6 characterize the boundary-layer and interior separations associated with the TCP flow.

2. COUETTE-POISEUILLE FLOW

2.1. Governing equations. We consider the viscous flow between two rotating coaxial cylinders with an axial pressure gradient. The instability of this flow was first discussed by S. Goldstein in 1933 [5], and a further investigation was made by S. Chandrasekhar [1].

Let r_1 and r_2 ($r_1 < r_2$) be the radii of two coaxial cylinders, Ω_1 and Ω_2 the angular velocities with which the inner and the outer cylinders rotate respectively, and

$$(2.1) \quad \mu = \Omega_2/\Omega_1, \quad \eta = r_1/r_2.$$

The hydrodynamical equations governing an incompressible viscous fluid between two coaxial cylinders are the Navier-Stokes equations in the cylinder polar coordinates (z, r, θ) , given by

$$(2.2) \quad \begin{cases} \frac{\partial u_z}{\partial t} + (u \cdot \nabla) u_z = \nu \Delta u_z - \frac{1}{\rho} \frac{\partial p}{\partial z}, \\ \frac{\partial u_r}{\partial t} + (u \cdot \nabla) u_r - \frac{u_\theta^2}{r} = \nu \left(\Delta u_r - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} - \frac{u_r}{r^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial r}, \\ \frac{\partial u_\theta}{\partial t} + (u \cdot \nabla) u_\theta + \frac{u_r u_\theta}{r} = \nu \left(\Delta u_\theta + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r^2} \right) - \frac{1}{r\rho} \frac{\partial p}{\partial \theta}, \\ \frac{\partial(r u_z)}{\partial z} + \frac{\partial(r u_r)}{\partial r} + \frac{\partial u_\theta}{\partial \theta} = 0, \end{cases}$$

where ν is the kinematic viscosity, ρ the density, $u = (u_r, u_\theta, u_z)$ the velocity field, p the pressure function, and

$$\begin{aligned} u \cdot \nabla &= u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r} + \frac{u_\theta}{r} \frac{\partial}{\partial \theta}, \\ \Delta &= \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \end{aligned}$$

In this article, we consider the case where a constant pressure gradient $\frac{\partial p}{\partial z} = p_0$ is applied in the z -direction, the cylinders rotate, and there are no radial motions. Under these conditions, (2.2) admits a steady state solution:

$$(2.3) \quad \mathcal{U}^{cp} = (u_z, u_r, u_\theta) = (W, 0, V), \quad p = p(r, z),$$

where

$$(2.4) \quad \frac{1}{\rho} \frac{dp}{dr} = V^2/r,$$

$$(2.5) \quad \nu \frac{d}{dr} \left(\frac{d}{dr} + \frac{1}{r} \right) V = 0,$$

$$(2.6) \quad \nu \left(\frac{d}{dr} + \frac{1}{r} \right) \frac{d}{dr} W = \frac{p_0}{\rho},$$

supplemented with the following boundary conditions

$$(2.7) \quad \begin{cases} (u_r, u_z) = \left(0, -\frac{p_0}{4\rho\nu} W_0 \right) & \text{at } r = r_1, r_2, \\ u_\theta = r_i \Omega_i & \text{at } r = r_i \text{ (i=1, 2)}. \end{cases}$$

Here $W_0 \geq 0$ is a constant.

We derive then from (2.4)—(2.7) that

$$(2.8) \quad V(r) = \frac{\Omega_1}{1-\eta^2} \left(r_1^2(1-\mu) \frac{1}{r} - (\eta^2 - \mu)r \right),$$

$$(2.9) \quad W(r) = -\frac{p_0}{4\rho\nu} \left(r_1^2 - r^2 + \frac{2r_1 d + d^2}{\ln(1 + d/r_1)} \ln \left(\frac{r}{r_1} \right) + W_0 \right),$$

$$(2.10) \quad p(r, z) = p_0 z + \rho \int \frac{1}{r} V^2(r) dr,$$

where $d (= r_2 - r_1)$ is the gap width, μ and η are as in (2.1).

Hereafter, we always assume that

$$(2.11) \quad \eta^2 > \mu,$$

which is a necessary condition for the instability to occur.

Classically, the flow described by $\mathcal{U}^c = (0, 0, V(r))$ is called the Couette flow, and the flow by $\mathcal{U}^p = (W(r), 0, 0)$ is called the Poiseuille flow. Therefore the flow described by (2.3) is a super-position of the Couette flow and the Poiseuille flow, and is usually called the Couette-Poiseuille (CP) flow.

The main objective of this article is to study the stability and transitions of the CP flow in both the physical and the phase spaces.

For this purpose, we consider the perturbed state

$$(2.12) \quad W + u_z, \quad u_r, \quad V + u_\theta, \quad p + p_0 z + \rho \int \frac{1}{r} V^2 dr.$$

Assume that the perturbation is axisymmetric and independent of θ , we derive from (2.2) that

$$(2.13) \quad \left\{ \begin{array}{l} \frac{\partial u_z}{\partial t} + (\tilde{u} \cdot \nabla) u_z = \nu \Delta u_z - W \frac{\partial u_z}{\partial z} - u_r \frac{dW}{dr} - \frac{1}{\rho} \frac{\partial p}{\partial z}, \\ \frac{\partial u_r}{\partial t} + (\tilde{u} \cdot \nabla) u_r - \frac{u_\theta^2}{r} = \nu \left(\Delta u_r - \frac{u_r}{r^2} \right) - \frac{1}{\rho} \frac{\partial p}{\partial r} \\ \quad - W \frac{\partial u_r}{\partial z} + \frac{2V}{r} r_\theta, \\ \frac{\partial u_\theta}{\partial t} + (\tilde{u} \cdot \nabla) u_\theta + \frac{u_\theta u_r}{r} = \nu \left(\Delta u_\theta - \frac{u_\theta}{r^2} \right) \\ \quad - \left(V' + \frac{1}{r} V \right) u_r - W \frac{\partial u_\theta}{\partial z}, \\ \frac{\partial(ru_z)}{\partial z} + \frac{\partial(ru_r)}{\partial r} = 0, \end{array} \right.$$

where $\tilde{u} = (u_z, u_r)$, and

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2},$$

$$(\tilde{u} \cdot \nabla) = u_z \frac{\partial}{\partial z} + u_r \frac{\partial}{\partial r}.$$

The spatial domain for (2.13) is $M = (0, L) \times (r_1, r_2)$, where L is the height of the cylinders. The physically sound boundary conditions are as follows

$$(2.14) \quad \left\{ \begin{array}{ll} u = (u_z, u_r, u_\theta) = 0 & \text{at } r = r_1, r_2, \\ u_z = 0, \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 & \text{at } z = 0, L. \end{array} \right.$$

2.2. Narrow-gap approximation. Consider the narrow-gap approximation with $\mu \geq 0$. Namely, we assume that the gap $d = r_2 - r_1$ is small in comparison to the mean radius:

$$(2.15) \quad d = r_2 - r_1 \ll \frac{1}{2}(r_2 + r_1).$$

Under this assumption, we can neglect the terms having the factors r^{-n} ($n \geq 1$) in the equations. Let

$$(2.16) \quad \alpha = \frac{\eta^2 - \mu}{1 - \eta^2}.$$

By (2.11), $\alpha > 0$. We derive then from (2.8) and (2.9) that

$$(2.17) \quad V' + \frac{1}{r} V = -2\alpha\Omega_1,$$

$$(2.18) \quad \frac{2V}{r} = 2\Omega_1 \left(1 - \frac{1-\mu}{1-\eta^2} \frac{r^2 - r_1^2}{r^2} \right) \simeq 2\Omega_1 \left(1 - \frac{(1-\mu)(r-r_1)}{d} \right),$$

$$(2.19) \quad W \simeq -\frac{p_0}{4\rho\nu} [(r-r_1)(r_2-r) + W_0].$$

Replacing u_θ by $\sqrt{\alpha} u_\theta$ in (2.13), we obtain the following approximation equations describing the flow between two cylinders with a narrow gap:

$$(2.20) \quad \begin{cases} \frac{\partial u_z}{\partial t} + (\tilde{u} \cdot \nabla) u_z = \nu \Delta u_z + \frac{p_0}{4\rho\nu} (d - 2\bar{r}) u_r \\ \quad + \frac{p_0}{4\rho\nu} (W_0 + \bar{r}(d - \bar{r})) \frac{\partial u_z}{\partial z} - \frac{1}{\rho} \frac{\partial p}{\partial z}, \\ \frac{\partial u_r}{\partial t} + (\tilde{u} \cdot \nabla) u_r = \nu \Delta u_r + 2\sqrt{\alpha} \Omega_1 \left(1 - \frac{(1-\mu)\bar{r}}{d} \right) u_\theta \\ \quad + \frac{p_0}{4\rho\nu} (W_0 + \bar{r}(d - \bar{r})) \frac{\partial u_r}{\partial z} - \frac{1}{p} \frac{\partial p}{\partial r}, \\ \frac{\partial u_\theta}{\partial t} + (\tilde{u} \cdot \nabla) u_\theta = \nu \Delta u_\theta + 2\sqrt{\alpha} \Omega_1 u_r + \frac{p_0}{4\rho\nu} (W_0 + \bar{r}(d - \bar{r})) \frac{\partial u_\theta}{\partial z}, \\ \text{div } \tilde{u} = 0, \end{cases}$$

where $\bar{r} = r - r_1$, $\Delta = \partial^2/\partial z^2 + \partial^2/\partial r^2$.

We need to consider the nondimensionalized form of (2.20). To this end, let

$$\begin{aligned} x &= x'd & (x = (z, r, r\theta)), \\ t &= t'd^2/\nu, \\ u &= u'\nu/d & (u = (u_z, u_r, u_\theta)), \\ p &= p'\rho\nu^2/d^2, \\ W_0 &= W'_0 d^2. \end{aligned}$$

Omitting the primes, the equations (2.20) can be rewritten as

$$(2.21) \quad \left\{ \begin{array}{l} \frac{\partial u_z}{\partial t} = \Delta u_z + \gamma(1 - 2\bar{r})u_r \\ \quad + \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_z}{\partial z} - \frac{\partial p}{\partial z} - (\tilde{u} \cdot \nabla)u_z, \\ \frac{\partial u_r}{\partial t} = \Delta u_r + \lambda(1 - (1 - \mu)\bar{r})u_\theta \\ \quad - \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_r}{\partial z} - \frac{\partial p}{\partial r} - (\tilde{u} \cdot \nabla)u_r, \\ \frac{\partial u_\theta}{\partial t} = \Delta u_\theta + \lambda u_r + \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_\theta}{\partial z} - (\tilde{u} \cdot \nabla)u_\theta, \\ \operatorname{div} \tilde{u} = 0, \end{array} \right.$$

where $\lambda = \sqrt{T}$. Here the Taylor number T and the nondimensional parameter γ are given by

$$(2.22) \quad T = \frac{4\alpha\Omega_1^2 d^4}{\nu^2}, \quad \gamma = \frac{p_0 d^3}{4\rho\nu^2}.$$

We remark here that the nondimensional parameter γ , proportional to the Reynolds number Re squared times the small gap d , is a small parameter.

When the gap $d = r_2 - r_1$ is small in comparison with the mean radius, we need not to distinguish the two equations (2.13) and (2.21) to discuss their dynamic bifurcation. Therefore, we always consider the problem (2.21) with (2.14) instead of (2.13) with (2.14). The equations (2.21) are supplemented with the following initial condition

$$(2.23) \quad u = \varphi \quad \text{at} \quad t = 0.$$

3. DYNAMIC BIFURCATION FOR PERTURBED SYSTEMS

3.1. Preliminaries. Let H_1 and H be two Hilbert spaces, and $H_1 \rightarrow H$ be a dense and compact inclusion. We consider the following nonlinear evolution equation:

$$(3.1) \quad \left\{ \begin{array}{l} \frac{du}{dt} = Au + \lambda Bu + \mathcal{L}_\lambda^\varepsilon u + G(u, \lambda), \\ u(0) = \varphi, \end{array} \right.$$

where $u : [0, \infty) \rightarrow H$ is the unknown function, $\lambda \in \mathbb{R}$ and $\varepsilon \in \mathbb{R}^m$ ($m \geq 1$) are the system parameters, and $A, B : H_1 \rightarrow H$ are linear operators, $\mathcal{L}_\lambda^\varepsilon : H_1 \rightarrow H$ are parameterized linear operators continuously depending on

$\lambda \in \mathbb{R}$, $\varepsilon \in \mathbb{R}^m$, which satisfy

$$(3.2) \quad \begin{cases} A : H_1 \rightarrow H & \text{a sectorial operator,} \\ A^\alpha : H_\alpha \rightarrow H & \text{the fractional power operators for } \alpha \in \mathbb{R}, \\ H_\alpha = D(A^\alpha) & \text{the domain of } A^\alpha \text{ with } H_0 = H; \end{cases}$$

$$(3.3) \quad \begin{cases} \mathcal{L}_\lambda^\varepsilon, B : H_\theta \rightarrow H & \text{bounded for some } \theta < 1, \\ \|\mathcal{L}_\lambda^\varepsilon\| \rightarrow 0 & \text{if } \varepsilon \rightarrow 0, \forall \lambda \in \mathbb{R}. \end{cases}$$

We know that $H_\alpha \subset H_\beta$ are compact inclusions for all $\alpha > \beta$, and the operators $A + \lambda B + \mathcal{L}_\lambda^\varepsilon : H_1 \rightarrow H$ are sectorial operators. Furthermore, we assume that the nonlinear terms $G(\cdot, \lambda) : H_\alpha \rightarrow H$ for some $\alpha < 1$ are a family of parameterized C^r bounded operators ($r \geq 1$), depending continuously on the parameter $\lambda \in \mathbb{R}$, such that

$$(3.4) \quad G(u, \lambda) = o(\|u\|_{H_\alpha}), \quad \forall \lambda \in \mathbb{R}.$$

Let $\{S(t)\}_{t \geq 0}$ be an operator semigroup generated by (3.1), which enjoys the semigroup properties. Then the solution of (3.1) can be expressed as $u(t, \varphi) = S(t)\varphi$ for any $t \geq 0$.

Definition 3.1. A set $\Sigma \subset H$ is called an invariant set of (3.1) if $S(t)\Sigma = \Sigma$ for any $t \geq 0$. An invariant set $\Sigma \subset H$ of (3.1) is called an attractor if Σ is compact, and there exists a neighborhood $U \subset H$ of Σ such that for any $\varphi \in U$ we have

$$\lim_{t \rightarrow \infty} \text{dist}(S(t)\varphi, \Sigma) = 0 \quad \text{in } H \text{ norm,}$$

and we say that Σ attracts U . Furthermore, if $U = H$, then Σ is called a global attractor of (3.1).

A number $\beta = \beta_1 + i\beta_2 \in \mathbb{C}$ is called an eigenvalue of a linear operator $L : H_1 \rightarrow H$ if there exist $x, y \in H_1$ with $x \neq 0$ such that

$$Lz = \beta z, \quad (z = x + iy).$$

The space

$$E_\beta = \bigcup_{n \in \mathbb{N}} \{x, y \in H_1 \mid (L - \beta)^n z = 0, z = x + iy\}$$

is called the eigenspace of L , and $x, y \in E_\beta$ are called eigenvectors of L .

Definition 3.2. A linear mapping $L^* : H_1 \rightarrow H$ is called the conjugate operator of $L : H_1 \rightarrow H$, if

$$\langle Lx, y \rangle_H = \langle x, L^*y \rangle_H, \quad \forall x, y \in H_1.$$

A linear operator $L : H_1 \rightarrow H$ is called symmetric if $L = L^*$.

3.2. Eigenvalue problem. Here we consider the eigenvalue problem for the perturbed linear operators

$$\begin{aligned} L_\lambda^\varepsilon &= L_\lambda + \mathcal{L}_\lambda^\varepsilon, \\ L_\lambda &= A + \lambda B. \end{aligned}$$

Let the eigenvalues (counting multiplicities) of L_λ and L_λ^ε be given respectively by

$$\begin{aligned} \{\beta_k(\lambda) \mid k = 1, 2, \dots\} &\subset \mathbb{C}, \\ \{\beta_k^\varepsilon(\lambda) \mid k = 1, 2, \dots\} &\subset \mathbb{C}. \end{aligned}$$

Suppose that the first eigenvalue $\beta_1(\lambda)$ of L_λ is simple near $\lambda = \lambda_0$, and the eigenvalues $\beta_k(\lambda)$ ($k = 1, 2, \dots$) satisfy the principle of exchange of stabilities (PES):

$$(3.5) \quad \beta_1(\lambda) = \begin{cases} < 0 & \text{if } \lambda < \lambda_0, \\ = 0 & \text{if } \lambda = \lambda_0, \\ > 0 & \text{if } \lambda > \lambda_0, \end{cases}$$

$$(3.6) \quad \operatorname{Re} \beta_j(\lambda_0) < 0 \quad \forall j \geq 2.$$

Let $e_\lambda, e_\lambda^* \in H_1$ be the eigenvectors of L_λ and L_λ^* corresponding to an eigenvalue $\beta(\lambda)$ respectively, i.e.,

$$(3.7) \quad \begin{cases} Ae_\lambda + \lambda Be_\lambda = \beta(\lambda)e_\lambda, \\ A^*e_\lambda^* + \lambda B^*e_\lambda^* = \beta(\lambda)e_\lambda^*. \end{cases}$$

We also assume that

$$(3.8) \quad \langle e_\lambda, e_\lambda^* \rangle_H = 1.$$

The following theorem is important for our discussion later, which provides a differential formula for simple eigenvalues of linear operators.

Theorem 3.3. *Let the eigenvalue $\beta(\lambda)$ in (3.7) be simple. Then $\beta(\lambda)$ is differentiable at λ , and with the assumption (3.8) we have*

$$(3.9) \quad \beta'(\lambda) = \langle Be_\lambda, e_\lambda^* \rangle_H.$$

Proof. By the genericity of simple eigenvalues of linear operators (see Kato [6] and Ma and Wang [11]), there is a number $\delta_0 > 0$ such that as $|\delta| < \delta_0$, $\beta(\lambda + \delta)$ is also a simple eigenvalue of $L_{\lambda+\delta}$. Let

$$(3.10) \quad \begin{cases} Au + (\lambda + \delta)Bu = \beta(\lambda + \delta)u, \\ u = e_\lambda + v_\delta, \\ \|v_\delta\|_H \longrightarrow 0 \quad \text{if } \delta \rightarrow 0. \end{cases}$$

We infer from (3.7), (3.8) and (3.10) that

$$\begin{aligned} &\langle (A + \lambda B)e_\lambda, e_\lambda^* \rangle_H + \langle (A + \lambda B)v_\delta, e_\lambda^* \rangle_H + \delta \langle Bu, e_\lambda^* \rangle_H \\ &= \beta(\lambda) + \beta(\lambda) \langle v_\delta, e_\lambda^* \rangle_H + \delta \langle Bu, e_\lambda^* \rangle_H \\ &= \beta(\lambda + \delta) + \beta(\lambda + \delta) \langle v_\delta, e_\lambda^* \rangle_H. \end{aligned}$$

Thus the last equality implies that

$$\frac{\beta(\lambda + \delta) - \beta(\lambda)}{\delta} (1 + \langle v_\delta, e_\lambda^* \rangle_H) = \langle B(e_\lambda + v_\delta), e_\lambda^* \rangle_H.$$

Then the theorem follows from (3.10). \square

Remark 3.4. For general linear operators $L_\lambda : H_1 \rightarrow H$, if L_λ are differentiable on λ with $\mathcal{L}_\lambda = \frac{d}{d\lambda} L_\lambda : H_1 \rightarrow H$ bounded, then Theorem 3.3 holds true as well, and

$$(3.11) \quad \beta'(\lambda) = \langle \mathcal{L}_\lambda e_\lambda, e_\lambda^* \rangle_H.$$

\square

By Theorem 3.3 or (3.11), for the perturbed linear operators L_λ^ε we immediately obtain the following corollary.

Corollary 3.5. *Assume the PES (3.5) and (3.6), and $\frac{d}{d\lambda} \mathcal{L}_\lambda^\varepsilon$ exists. Then for each $|\varepsilon| < \delta$ for some $\delta > 0$, there exists λ_0^ε , with $\lim_{\varepsilon \rightarrow 0} \lambda_0^\varepsilon = \lambda_0$, such that the eigenvalues $\beta_k^\varepsilon(\lambda)$ of L_λ^ε at λ_0^ε satisfy the following PES:*

$$(3.12) \quad \beta_1^\varepsilon(\lambda) \begin{cases} < 0 & \text{if } \lambda < \lambda_0^\varepsilon, \\ = 0 & \text{if } \lambda = \lambda_0^\varepsilon, \\ > 0 & \text{if } \lambda > \lambda_0^\varepsilon, \end{cases}$$

$$(3.13) \quad \operatorname{Re} \beta_j^\varepsilon(\lambda_0^\varepsilon) < 0, \quad \forall 2 \leq j.$$

Moreover, $\beta_1^\varepsilon(\lambda)$ has the following expansion at $\lambda = \lambda_0^\varepsilon$:

$$(3.14) \quad \begin{cases} \beta_1^\varepsilon(\lambda) = \alpha_\varepsilon(\lambda - \lambda_0^\varepsilon) + o(|\lambda - \lambda_0^\varepsilon|), \\ \alpha_\varepsilon = \langle (B + \frac{d}{d\lambda} \mathcal{L}_{\lambda_0^\varepsilon}^\varepsilon) e_\varepsilon, e_\varepsilon^* \rangle_H, \end{cases}$$

where e_ε and e_ε^* satisfy

$$(3.15) \quad L_{\lambda_0^\varepsilon}^\varepsilon e_\varepsilon = 0, \quad L_{\lambda_0^\varepsilon}^{\varepsilon*} e_\varepsilon^* = 0.$$

3.3. Bifurcation for perturbed equations. We study now the dynamic bifurcation for the nonlinear evolution equation (3.1), which is a perturbation of the following equation

$$(3.16) \quad \frac{du}{dt} = Au + \lambda Bu + G(u, \lambda).$$

Under the conditions (3.7) and (3.8), let $e_0 \in H_1$ be an eigenvector of L_λ at $\lambda = \lambda_0$:

$$(3.17) \quad Ae_0 + \lambda_0 Be_0 = 0, \quad \|e_0\| = 1.$$

We assume that

$$(3.18) \quad \begin{cases} L_\lambda = A + \lambda B & \text{are symmetric,} \\ G(u, \lambda) & \text{are bilinear,} \\ \langle G(u, v), v \rangle_H = 0 & \forall u, v \in H_\alpha. \end{cases}$$

Here if G is bilinear, one can write G as $G(\cdot, \cdot)$, which is linear for each argument.

By Theorem 6.15 and Remark 6.8 in [11], together with Corollary 3.5 above, we obtain immediately the following theorem.

Theorem 3.6. *Assume that (3.2)—(3.6) and (3.18) hold true, and $u = 0$ is a globally asymptotically stable equilibrium point of (3.16) at $\lambda = \lambda_0$. Then there are constants $\delta_1 > 0$ and $\delta_2 > 0$ such that if $|\varepsilon| < \delta_1$ and $0 < \lambda - \lambda_0^\varepsilon < \delta_2$, where λ_0^ε is given in (3.12) and (3.13), then the following assertions hold true.*

- (1) *There exists an attractor $\Sigma_\lambda^\varepsilon = \{u_1^\lambda, u_2^\lambda\} \subset H$ of (3.1), where u_1^λ and u_2^λ are steady states given by*

$$(3.19) \quad \begin{cases} u_1^\lambda = \sigma_1(\lambda, \varepsilon)e_0 + v_1(\lambda, \varepsilon), \\ u_2^\lambda = -\sigma_2(\lambda, \varepsilon)e_0 + v_2(\lambda, \varepsilon), \\ v_i(\lambda, \varepsilon) = o(|\sigma_i(\lambda, \varepsilon)|) \in H_1, \quad i = 1, 2. \end{cases}$$

Here $\sigma_i(\lambda, \varepsilon) \in \mathbb{R}$ can be expressed as

$$(3.20) \quad \begin{cases} \sigma_{1,2} = \frac{\sqrt{b^2(\varepsilon) + 4\beta_1^\varepsilon(\lambda)} \mp |b(\varepsilon)|}{2C} + o(|b|, |\beta_1^\varepsilon|), \\ b(\varepsilon) = \langle G(e_\varepsilon, \lambda_0^\varepsilon), e_\varepsilon^* \rangle_H, \end{cases}$$

where $e_\varepsilon, e_\varepsilon^*$ are as in (3.15), $\beta_1^\varepsilon(\lambda)$ as in (3.14), and $C > 0$ is a constant.

- (2) $\Sigma_\lambda^\varepsilon$ attracts every bounded open set in $H \setminus \Gamma$, where Γ is the stable manifold of $u = 0$ with co-dimension one. Namely, H can be decomposed into two open sets U_1^λ and U_2^λ :

$$H = \overline{U_1^\lambda} + \overline{U_2^\lambda}, \quad U_1^\lambda \cap U_2^\lambda = \emptyset, \quad 0 \in \Gamma = \partial U_1^\lambda \cap \partial U_2^\lambda,$$

with $u_i^\lambda \in U_i^\lambda$ ($i = 1, 2$), such that

$$\lim_{t \rightarrow \infty} \|u(t, \varphi) - u_i^\lambda\|_H = 0, \quad \text{if } \varphi \in U_i^\lambda \text{ } (i = 1, 2),$$

where $u(t, \varphi)$ is the solution of (3.1).

A few remarks are now in order.

Remark 3.7. It is clear that the eigenvectors in (3.15) satisfy that $e_\varepsilon \rightarrow e_0$, $e_\varepsilon^* \rightarrow e_0$ when $\varepsilon \rightarrow 0$, where e_0 is as in (3.17). Therefore, by (3.18) we have $b(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Remark 3.8. For a given $\varepsilon > 0$, if $b(\varepsilon) \neq 0$, then the equation (3.1) has a saddle-node bifurcation at $\lambda^* < \lambda_0^\varepsilon$ as shown in Figure 3.1(a); if $b(\varepsilon) = 0$, then (3.1) has an (supercritical) attractor bifurcation as shown in Figure 3.1(b).

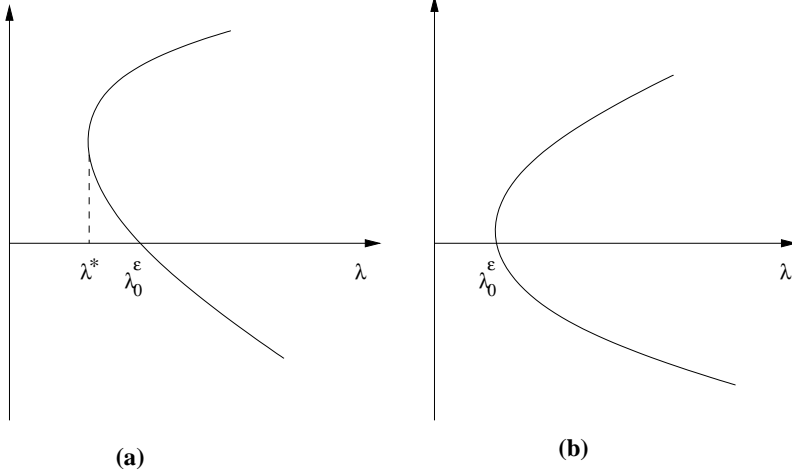


FIGURE 3.1. (a) Saddle-node bifurcation and hysteresis, and (b) supercritical attractor bifurcation.

To apply the above dynamic bifurcation theorem it is crucial to verify the global asymptotic stability for (3.16) at $\lambda = \lambda_0$. The following theorem plays an important role to ensure this condition, which was proved in [9, 11].

Theorem 3.9. *Under the conditions (3.5), (3.6) and (3.18), if $G(e_0, \lambda_0) \neq 0$, then $u = 0$ is a globally asymptotically stable equilibrium point of (3.16) at $\lambda = \lambda_0$.*

4. DYNAMIC BIFURCATION AND STABILITY OF THE COUETTE-POISEUILLE FLOW

4.1. Functional setting. We recall here the functional setting and some basic mathematical properties of equations (2.21) with boundary conditions given by (2.14). The spatial domain is $M = (r_1, r_1 + 1) \times (0, L)$, the coordinate system is (r, z) , the velocity field is $u = (u_z, u_r, u_\theta)$ and $\tilde{u} = (u_z, u_r)$. Let

$$(4.1) \quad \begin{cases} H = \{u = (\tilde{u}, u_\theta) \in L^2(M)^3 \mid \operatorname{div} \tilde{u} = 0, \tilde{u} \cdot n|_{\partial M} = 0\}, \\ V = \{u \in H^1(M)^3 \cap H \mid u = 0 \text{ at } r = r_1, r_2, u_z = 0 \text{ at } z = 0, L\}, \\ H_1 = \{u \in H^2(M)^3 \cap V \mid \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 \text{ at } z = 0, L\}, \end{cases}$$

where $H^k(M)$ is the usual Sobolev spaces.

The equations (2.21) are two-dimensional, therefore the results concerning the existence and regularity of a solution for (2.21) with (2.14) are classical. For each initial value $u_0 \in H$, (2.21) and (2.23) with (2.14) has a weak solution

$$u \in L^\infty([0, T], H) \cap L^2([0, T], V), \quad \forall T > 0.$$

If $u_0 \in V$, (2.21), (2.23) with (2.14) has a unique solution

$$u \in C([0, T], V) \cap L^2([0, T], H_1), \quad \forall T > 0.$$

For every $u_0 \in H$ and $k \geq 1$ there exists a $\tau_0 > 0$ such that the solution $u(t, u_0)$ of (2.21), (2.23) with (2.14) satisfies

$$u(t, u_0) \in H^k(M)^3, \quad \forall t > \tau_0.$$

Furthermore, for each $k \geq 1$ and $u_0 \in H^k(M)^3$, there exists a number $C > 0$ depending on k and φ which bounds the solution $u(t, u_0)$ of (2.21) and (2.23) with (2.14) in the H^k -norm (see [2]):

$$\|u(t, u_0)\|_{H^k} \leq C, \quad \forall t \geq 0.$$

Finally, it is essentially known that for any λ, γ and μ the equations (2.21) with (2.14) have a global attractor; see Temam [14] and Foias *et al.* [2].

4.2. Eigenvalue problem of the linearized equations. The linearized equations of (2.21) with (2.14) are given by

$$(4.2) \quad \begin{cases} -\Delta u_z - \gamma(1 - 2\bar{r})u_r + \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_z}{\partial z} + \frac{\partial p}{\partial z} = 0, \\ -\Delta u_r + \lambda(1 - \mu)\bar{r}u_\theta + \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_r}{\partial z} + \frac{\partial p}{\partial r} = \lambda u_\theta, \\ -\Delta u_\theta - \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_\theta}{\partial z} = \lambda u_r, \\ \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} = 0, \end{cases}$$

with the boundary conditions

$$(4.3) \quad \begin{cases} u = 0 & \text{at } r = r_1, r_1 + 1, \\ u_z = 0, \quad \frac{\partial u_r}{\partial z} = \frac{\partial u_\theta}{\partial z} = 0 & \text{at } z = 0, L, \end{cases}$$

where $\bar{r} = r - r_1$, $\lambda = \sqrt{T}$, T is the Taylor number, T and γ are given by (2.22), and μ is defined by (2.1). We denote

$$\varepsilon = (\gamma, 1 - \mu) \in (0, \infty) \times (0, \infty).$$

Let $\lambda_0^\varepsilon > 0$ be the first eigenvalue of (4.2), (4.3), and we call

$$(4.4) \quad T_c = (\lambda_0^\varepsilon)^2$$

the critical Taylor number.

When $\varepsilon \rightarrow 0$, the system (4.2) reduces to the following symmetric linear equations

$$(4.5) \quad \begin{cases} -\Delta u_z + \frac{\partial p}{\partial z} = 0, \\ -\Delta u_r + \frac{\partial p}{\partial r} = \lambda u_\theta, \\ -\Delta u_\theta = \lambda u_r, \\ \frac{\partial u_z}{\partial z} + \frac{\partial u_r}{\partial r} = 0. \end{cases}$$

It is known that the eigenvalue problem (4.5) with boundary conditions (4.3) has the eigenvectors as follows

$$(4.6) \quad u_0 = (u_z, u_r, u_\theta) \\ = \left(\frac{1}{a} \sin az R'(r), \frac{1}{a} \sin az R'(r), \frac{-1}{a^2 \lambda_0} \cos az \left(\frac{d^2}{dz^2} - a^2 \right)^2 R(r) \right),$$

where a is the wave length which is given by (see [1])

$$a = \frac{K\pi}{L},$$

for some $K \geq 1$, with K depending on L . When L is large, K is taken to be the minimizer of

$$\min_k \left| \frac{k\pi}{L} - 3.117 \right|.$$

The number λ_0 is the first eigenvalue of (4.5), and when $a \simeq 3.117$, $\lambda_0^2 \simeq 1700$.

Consider

$$\begin{cases} \left(\frac{d^2}{dr^2} - a^2 \right)^3 R = -a^2 \lambda_0^2 R, \\ R = 0, \quad R' = 0, \quad \left(\frac{d^2}{dr^2} - a^2 \right)^2 R = 0 \text{ at } r = r_1, r_1 + 1. \end{cases}$$

Its solution can be expressed as (see Chapter II-15 of [1])

$$(4.7) \quad R(r) = \cos \alpha_0 x - \beta_1 \cosh \alpha_1 x + \cos \alpha_2 x + \beta_2 \sinh \alpha_1 x \sin \alpha_2 x,$$

where $x = \bar{r} - \frac{1}{2}$, $\bar{r} = r - r_1$,

$$(4.8) \quad \begin{cases} \beta_1 = 0.06151664, & \beta_2 = 0.10388700, \\ \alpha_0 = 3.973639, & \alpha_1 = 5.195214, & \alpha_2 = 2.126096. \end{cases}$$

In addition, the first eigenvalue λ_0 of (4.5) with (4.3) is simple except for the following values of L :

$$(4.9) \quad L_k = \frac{k\pi}{\alpha_0}, \quad k = 1, 2, \dots, \quad a_0 \simeq 4.2.$$

4.3. Secondary flows and their stability. We are now in position to state and prove the main theorem in this section. We remark here that under conditions (2.11) and (2.15) with scaling $d = 1$, the condition $\mu \rightarrow 1$ can be replaced by

$$(4.10) \quad (1 - \mu)r_1 = 2 + \delta$$

for some $\delta > 0$ and r_1 sufficiently large. This condition implies that

$$\alpha = (\eta^2 - \mu)/(1 - \eta^2) \simeq \delta.$$

Theorem 4.1. *Assume that (4.10) holds true, and $L \neq L_k$ is sufficiently large, where L_k is given by (4.9). Then there are $R > 0$ and $\delta_1 > 0$ such that for any $r_1 > R$ and $|\gamma| < \delta_1$, the following assertions hold true, provided that the Taylor number $T = \lambda^2$ satisfies that $0 < T - T_c < \delta_2$ with T_c as in (4.4) for some $\delta_2 > 0$, or equivalently $0 < \lambda - \lambda_0^\varepsilon < \tilde{\delta}_2$ for some $\tilde{\delta}_2 > 0$:*

- (1) *There exists an attractor $\Sigma_\lambda^\varepsilon = \{u_1^\lambda, u_2^\lambda\} \subset H$ of (2.21) with (2.14), where u_1^λ and u_2^λ are steady state solutions given by*

$$(4.11) \quad \begin{cases} u_1^\lambda = \sigma_1(\lambda, \varepsilon)u_0 + v_1(\lambda, \varepsilon), \\ u_2^\lambda = -\sigma_2(\lambda, \varepsilon)u_0 + v_2(\lambda, \varepsilon), \\ v_i(\lambda, \varepsilon) = o(|\sigma_i(\lambda, \varepsilon)|), \quad i = 1, 2. \end{cases}$$

Here $u_0 = (u_z, u_r, u_\theta)$ is the eigenfunction given by (4.6), and

$$(4.12) \quad \begin{cases} \sigma_{1,2} = \frac{\sqrt{b^2(\varepsilon) + 4\beta_1^\varepsilon(\lambda)} \mp |b(\varepsilon)|}{2C} + o(|b|, |\beta_1^\varepsilon|), \\ \beta_1^\varepsilon(\lambda) = \alpha(\varepsilon)(\lambda - \lambda_0^\varepsilon) + o(|\lambda - \lambda_0^\varepsilon|), \quad \alpha(0) > 0, \\ b(\varepsilon) = - \int_M (\tilde{w}^\varepsilon \cdot \nabla) w^\varepsilon \cdot w^{\varepsilon*} dx, \end{cases}$$

where $w^\varepsilon = (\tilde{w}^\varepsilon, w_\theta^\varepsilon)$, with $\tilde{w}^\varepsilon = (w_z^\varepsilon, w_r^\varepsilon)$, is the eigenfunction of (4.2) and (4.3) corresponding to λ_0^ε , $w^{\varepsilon*}$ is the dual eigenfunction, and $C > 0$ a constant.

- (2) *The attractor $\Sigma_\lambda^\varepsilon$ attracts all bounded open sets of $H \setminus \Gamma$, where Γ is the stable manifold of $u = 0$ with co-dimension one. Namely, the space H can be decomposed into two open sets U_1^λ and U_2^λ :*

$$H = \overline{U_1^\lambda} + \overline{U_2^\lambda}, \quad U_1^\lambda \cap \overline{U_2^\lambda} = \emptyset, \quad 0 \in \Gamma = \partial U_1^\lambda \cap \partial U_2^\lambda,$$

with $u_i^\lambda \in U_i^\lambda$ ($i = 1, 2$), such that

$$\lim_{t \rightarrow \infty} \|u(t, \varphi) - u_i^\lambda\|_{L^2} = 0, \quad \text{when } \varphi \in U_i^\lambda \quad (i = 1, 2),$$

where $u(t, \varphi)$ is the solution of (2.21) with (2.15) and (2.23).

Remark 4.2. As mentioned in the Introduction, the existence of a bifurcation to a steady state solution is classical, and can be proved by the classical Krasnoselskii theorem. There are two new ingredients in this theorem. First, here it is proved that the basin of attraction of the bifurcated attractor $\Sigma_\lambda^\varepsilon$ is $H \setminus \Gamma$. Second the explicit form of the bifurcated solutions. These new

ingredients enable us to determine, in the next two sections, the asymptotic structure in the physical space of the solutions of the problem, leading to justification of the flow structure of the TCP flow.

Proof of Theorem 4.1. We shall apply Theorem 3.6 to prove this theorem. Let H and H_1 be defined by (4.1). We define the mappings

$$(4.13) \quad \begin{cases} L_\lambda = A + \lambda B : H_1 \rightarrow H, \\ L_\lambda^\varepsilon = L_\lambda + \mathcal{L}_\lambda^\varepsilon : H_1 \rightarrow H, \\ G : H_1 \rightarrow H, \end{cases}$$

by

$$\begin{aligned} Au &= \{P\Delta\tilde{u}, \Delta u_\theta\}, \\ Bu &= \{P(0, u_\theta), u_r\}, \\ \mathcal{L}_\lambda^\varepsilon u &= \left\{ P \left(\gamma(1 - 2\bar{r})u_r + \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_z}{\partial z}, \lambda(1 - \mu)\bar{r}u_\theta \right. \right. \\ &\quad \left. \left. + \gamma(W_0 + \bar{r}(1 - \bar{r}(1 - \bar{r})))\frac{\partial u_r}{\partial z} \right), \gamma(W_0 + \bar{r}(1 - \bar{r}))\frac{\partial u_\theta}{\partial z} \right\}, \\ G(u) &= \{-P(\tilde{u} \cdot \nabla)\tilde{u}, -(\tilde{u} \cdot \nabla)u_\theta\}. \end{aligned}$$

where $u = (u_z, u_r, u_\theta) \in H_1$, and the operator $P : L^2(M)^3 \rightarrow H$ is the Leray projection. Thus the equation (2.21) with (2.15) can be written in the abstract form

$$\frac{du}{dt} = L_\lambda u + \mathcal{L}_\lambda^\varepsilon u + G(u).$$

It is well known that the above defined operators satisfy the conditions (3.2)-(3.4). In particular

$$\begin{aligned} B : H_\alpha &\longrightarrow H \text{ is bounded for all } \alpha \geq 0, \\ \mathcal{L}_\lambda^\varepsilon : H_\alpha &\longrightarrow H \text{ is bounded for all } \alpha \geq \frac{1}{2}, \\ G : H_\alpha &\longrightarrow H \text{ is analytic for } \alpha > \frac{1}{2}, \end{aligned}$$

where H_α is the fractional Sobolev spaces defined by the interpolation between $H_0 = H$ and H_1 .

It is clear that L_λ is symmetric, and $G : H_\alpha \rightarrow H$ ($\alpha > \frac{1}{2}$) is bilinear satisfying that

$$\langle G(u, v), v \rangle_H = 0$$

Hence, the condition (3.18) holds true.

With the operators in (4.13), the eigenvalue equation (3.17) corresponds to the problem (4.5) with (4.3), and the eigenvector e_0 is precisely the $u_0 = (u_z, u_r, u_\theta)$ given by (4.6).

Now consider the eigenvalue problem

$$(4.14) \quad L_\lambda u = \beta(\lambda)u, \quad u = (u_z, u_r, u_\theta) \in H_1.$$

By (4.13), the abstract form (4.14) is equivalent to the following eigenvalue equations in H_1 :

$$(4.15) \quad \begin{cases} \Delta u_z - \frac{\partial p}{\partial z} = \beta(\lambda) u_z, \\ \Delta u_r - \frac{\partial p}{\partial r} + \lambda u_\theta = \beta(\lambda) u_r, \\ \Delta u_\theta + \lambda u_r = \beta(\lambda) u_\theta, \\ \operatorname{div} \tilde{u} = 0. \end{cases}$$

It is known that the eigenvalues β_k ($k = 1, 2, \dots$) of (4.15) in H_1 are real numbers satisfying

$$(4.16) \quad \begin{cases} \beta_1(\lambda) \geq \beta_2(\lambda) \geq \dots \geq \beta_k(\lambda) \geq \dots, \\ \beta_k \longrightarrow -\infty \quad \text{as } k \rightarrow \infty. \end{cases}$$

Because the first eigenvalue λ_0 of (4.5) with (4.3) is simple, the first eigenvalue $\beta_1(\lambda)$ of (4.15) with (4.3) is also simple at $\lambda = \lambda_0$. Noting that for the first eigenfunction u_0 of (4.15) with (4.3) at $\lambda = \lambda_0$, which is given by (4.6), we have

$$\begin{aligned} \langle Bu_0, u_0 \rangle_H &= 2 \int_M u_r u_\theta dx \\ &= \frac{2}{a^2 \lambda_0} \int_0^L \cos^2 az dz \int_{r_1}^{r_1+1} R \left(\frac{d^2}{dz^2} - a^2 \right)^2 R dr \\ &= \frac{2}{a^2 \lambda_0} \int_0^L \cos^2 az dz \int_{r_1}^{r_1+1} \left| \left(\frac{d^2}{dz^2} - a^2 \right) R \right|^2 dr \\ &> 0. \end{aligned}$$

Hence, from Theorem 3.3 and (4.16) we can derive the conditions (3.5) and (3.6) at $\lambda = \lambda_0$.

Finally, by direct computation, we have

$$G(u_0) \neq 0,$$

for the first eigenfunction u_0 of (4.5) with (4.3). By Theorem 3.9 one can derive that $u = 0$ is a globally asymptotically stable steady state solution of the following equations with (4.3) at $\lambda = \lambda_0$:

$$\begin{aligned} \frac{\partial u_z}{\partial t} &= \Delta u_z - \frac{\partial p}{\partial z} - (\tilde{u} \cdot \nabla) u_z, \\ \frac{\partial u_r}{\partial t} &= \Delta u_r + \lambda u_\theta - \frac{\partial p}{\partial r} - (\tilde{u} \cdot \nabla) u_r, \\ \frac{\partial u_\theta}{\partial t} &= \Delta u_\theta + \lambda u_r - (\tilde{u} \cdot \nabla) u_\theta, \\ \operatorname{div} \tilde{u} &= 0, \end{aligned}$$

which correspond to the abstract equation (3.16). Thus, this theorem follows from Theorem 3.6. The proof is complete.

5. STRUCTURAL TRANSITION OF THE COUETTE-POISEUILLE FLOW: BOUNDARY-LAYER SEPARATION

In this and the next sections, we study the structure and its transitions in the CP and TCP flows, using a recently developed geometric theory of 2D incompressible flows by the authors; see a recent monograph by the authors [12] and the references therein. The results obtained provide a rigorous characterization on how, when and where the Taylor vortices are generated.

5.1. Geometric Theory of Incompressible Flows. We first give a recapitulation of some results in the geometric theory of incompressible flows, which will be used in characterizing the structure and its transitions in the CP and TCP flows. We refer interested readers to above references for further details of the theory.

5.1.1. Structural stability. Let $M \subset \mathbb{R}^2$ be an open set, and $C^r(M, \mathbb{R}^2)$ be the space of all C^r ($r \geq 1$) vector field on M . We denote

$$\begin{aligned} D^r(M, \mathbb{R}^2) &= \{v \in C^r(M, \mathbb{R}^2) \mid \operatorname{div} v = 0, v \cdot n|_{\partial\Omega} = 0\}, \\ B^r(M, \mathbb{R}^2) &= \left\{v \in D^r(M, \mathbb{R}^2) \mid \frac{\partial v_n}{\partial \tau} = 0 \text{ on } \partial\Omega\right\}, \\ B_0^r(M, \mathbb{R}^2) &= \{v \in D^r(M, \mathbb{R}^2) \mid v = 0 \text{ on } \partial\Omega\}, \end{aligned}$$

where τ, n are the unit tangent and normal vectors on ∂M .

Let $X = D^r(M, \mathbb{R}^2)$, or $B^r(M, \mathbb{R}^2)$, or $B_0^r(M, \mathbb{R}^2)$ in the following definitions.

Definition 5.1. *Two vector fields $u, v \in X$ are called topologically equivalent if there exists a homeomorphism of $\varphi : M \rightarrow M$, which takes the orbits of u to orbits of v and preserves their orientation.*

Definition 5.2. *A vector field $u \in X$ is called structurally stable in X if there exists a neighborhood $O \subset X$ of u such that for any $v \in O$, u and v are topologically equivalent.*

For $u \in B_0^r(M, \mathbb{R}^2)$ ($r \geq 2$), a different singularity concept for points on the boundary was introduced in [7], we proceed as follows.

- (1) A point $p \in \partial M$ is called a ∂ -regular point of u if

$$\frac{\partial u_\tau(p)}{\partial n} \neq 0,$$

and otherwise $p \in \partial M$ is called a ∂ -singular point.

- (2) A ∂ -singular point $p \in \partial M$ of u is called nondegenerate if

$$\det \begin{pmatrix} \frac{\partial^2 u_\tau(p)}{\partial \tau \partial n} & \frac{\partial^2 u_\tau(p)}{\partial n^2} \\ \frac{\partial^2 u_n(p)}{\partial \tau \partial n} & \frac{\partial^2 u_n(p)}{\partial n^2} \end{pmatrix} \neq 0.$$

A nondegenerate ∂ -singular point of u is also called a ∂ -saddle point.

- (3) A vector field $u \in B_0^r(M, \mathbb{R}^2)$ ($r \geq 2$) is called D -regular if u is regular in M , and all ∂ -singular points of u on ∂M are nondegenerate.

The following theorem provides necessary and sufficient conditions for structural stability in $B_0^r(M, \mathbb{R}^2)$. For the structural stability theorems in $D^r(M, \mathbb{R}^2)$ and $B^r(M, \mathbb{R}^2)$, see [12, 8].

Theorem 5.3 (Ma and Wang [7]). *Let $u \in B_0^r(M, \mathbb{R}^2)$ ($r \geq 2$). Then u is structurally stable in $B_0^r(M, \mathbb{R}^2)$ if and only if*

- (1) u is D -regular,
- (2) all interior saddle points of u are self-connected; and
- (3) each ∂ -saddle point of u on ∂M is connected to a ∂ -saddle point on the same connected component of ∂M .

Moreover, the set of all structurally stable vector fields is open and dense in $B_0^r(M, \mathbb{R}^2)$.

5.1.2. Structural bifurcation and boundary-layer separation. Let $u(\cdot, \lambda) \in B_0^r(M, \mathbb{R}^2)$ be a one-parameter family of vector fields. Assume that the boundary ∂M contains a flat part $\Gamma \subset \partial M$ and $x_0 \in \Gamma$. For simplicity, we take a coordinate system (x_1, x_2) with x_0 at the origin and with Γ given by

$$\Gamma = \{(x_1, 0) \mid |x_1| < \delta \text{ for some } \delta > 0\}.$$

Obviously, the tangent and normal vectors on Γ are the unit vectors in the x_1 - and x_2 -directions respectively. In a neighborhood of $x_0 \in \Gamma$, $u(x, \lambda)$ can be expressed near $x = 0$ by

$$u(x, \lambda) = x_2 v(x, \lambda).$$

To proceed, we consider the Taylor expansion of $u(x, \lambda)$ at $\lambda = \lambda_0$:

$$(5.1) \quad \begin{cases} u(x, \lambda) = u^0(x) + (\lambda - \lambda_0)u^1(x) + o(|\lambda - \lambda_0|), \\ u^0(x) = x_2 v^0(x). \end{cases}$$

Let $u^0 = (u_1^0, u_2^0)$. We assume that

$$(5.2) \quad \frac{\partial u^0(0)}{\partial n} = v^0(0) = 0,$$

$$(5.3) \quad \text{ind}(v^0, 0) = 0,$$

$$(5.4) \quad \frac{\partial u^1(0)}{\partial n} \neq 0,$$

$$(5.5) \quad \frac{\partial^{k+1} u_1^0(0)}{\partial^k \tau \partial n} \neq 0, \quad \text{for some } k \geq 2.$$

The following is a theorem describing the structural bifurcation on boundary for the case with index zero, which characterizes the boundary-layer separation for 2- D incompressible fluid flow.

Theorem 5.4. *Let $u(\cdot, \lambda) \in B_0^r(M, \mathbb{R}^2)$ be as given in (5.1) satisfying the conditions (5.2)-(5.5). Then the following assertions hold true:*

- (1) As $\lambda < \lambda_0$ (or $\lambda > \lambda_0$), the flow described by $u(x, \lambda)$ is topologically equivalent to a parallel flow near $x_0 \in \Gamma$, as shown in Figure 5.1a.
- (2) As $\lambda_0 < \lambda$ (or $\lambda > \lambda_0$), there are some closed orbits of u separated from $x_0 \in \Gamma$, i.e. some vortices separated from $x_0 \in \Gamma$, as shown schematically in either Figure 5.1 (c) or (d).
- (3) If $k = 2$ in (5.5), the vortex separated from $x_0 \in \Gamma$ is unique, and the flow structure enjoys the following properties:
 - (a) there are exactly two ∂ -saddle points $x^\pm = (x_1^\pm, 0) \in \Gamma$ of $u(x, \lambda)$ near $x_0 = 0$ with $x_1^- < 0 < x_1^+$, which are connected by an interior orbit $\gamma(\lambda)$ of $u(x, \lambda)$;
 - (b) the closed orbits separated from x_0 are enclosed by the interior orbit $\gamma(\lambda)$ and the portion of the boundary between x^- and x^+ ;
 - (c) the interior orbit $\gamma(\lambda)$ shrinks to x_0 as $\lambda \rightarrow \lambda_0$.

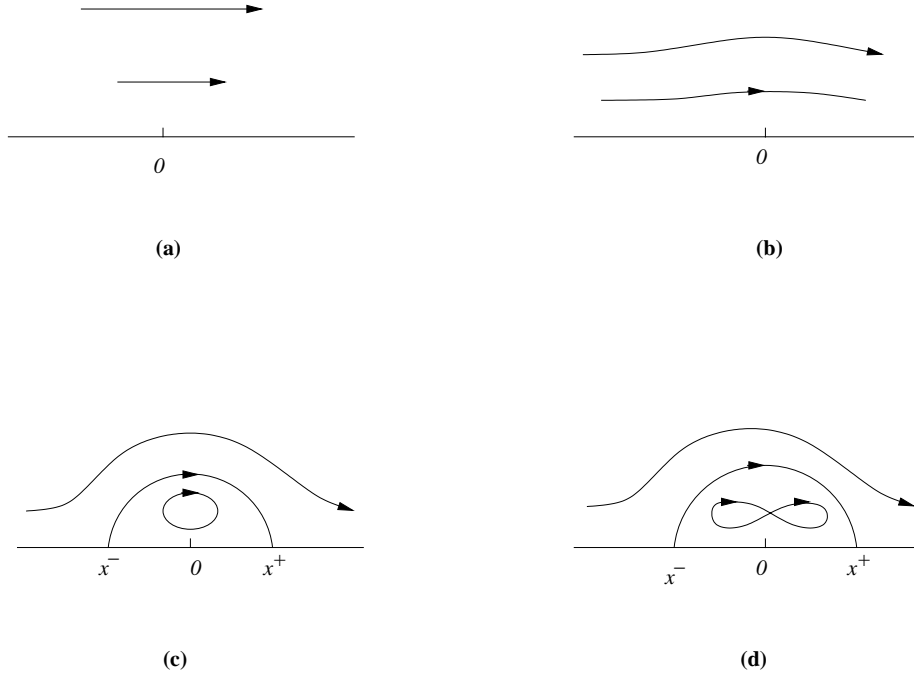


FIGURE 5.1. Boundary-layer separation and re-attachment near a flat boundary.

We mention that when ∂M is curved at $x_0 \in \partial M$, corresponding theorem is also true; see [12, 3, 4].

5.2. Boundary-layer separation in the TCP flow. Hereafter, we always assume that the conditions in Theorem 4.1 hold true. The main objective of this and the next sections is to study the asymptotic structure and its transition in the physical space in the CP flow and the TCP flow regimes.

As mentioned in the Introduction, the results obtained in these two sections provide a rigorous characterization on how, when and where the Taylor vortices are generated. In particular, contrary to what is commonly believed, we show that the propagating Taylor vortices (PTV) do not appear after the first dynamical bifurcation, and they appear only after further increase the Taylor number so that the boundary-layer or interior separation occurs.

We shall prove that the type of separations (boundary-layer or interior) is dictated by the (vertical) through flow driven by a pressure gradient along the rotating axis. Hence we consider two cases. The first is the case where through flow vanishes at the cylinder walls, i.e. $W_0 = 0$ in (2.7). This leads to boundary-layer separation, and will be studied in this section.

The second is the case where $W_0 \neq 0$, leading to interior separations. This case will be addressed in the next section.

5.2.1. Main results. Consider now the case where $W_0 = 0$, and the main results are the following two theorems. The first gives a precise characterization on how, when and where the propagating Taylor vortices (PTV) are generated for the bifurcated solutions, and the second theorem describes time evolution to the PTV for the solutions of the equations.

Theorem 5.5. *Assume that the conditions in Theorem 4.1 hold true, and assume that the constant velocity in boundary condition (2.7) vanishes, i.e., $W_0 = 0$. Then there is a $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$ where γ is defined by (2.22), the following assertions hold true for the two (bifurcated) steady state solutions $v_i^\lambda = \mathcal{U}^{cp} + u_i^\lambda$ ($i = 1, 2$) of (2.2) and (2.7), where \mathcal{U}^{cp} and u_i^λ are given by (2.3) and (4.11):*

- (1) For v_1^λ , there is a number $\lambda_1 > \lambda_0^\varepsilon$ (λ_0^ε as in (4.4)) such that
 - (a) for $\lambda_0^\varepsilon < \lambda < \lambda_1$, the vector field $\tilde{v}_1^\lambda = (W + u_{1z}^\lambda, u_{1r}^\lambda)$ is topologically equivalent to the vertical shear flow $(W, 0)$ as shown in Figure 5.2, and
 - (b) for $\lambda_1 < \lambda$, there is a unique center of \tilde{v}_1^λ separated from each point $(z_n, r_1) \in \partial M$, where $z_n \simeq (4n + 1)\pi/2a$ (resp. $(\bar{z}_n, r_2) \in \partial M$, $\bar{z}_n = (4n + 3)\pi/2a$) for $n = 0, 1, \dots, k_0$, as shown in Figure 5.3, with a as in (4.6).
- (2) For v_2^λ , there is a number $b_0 > 0$ such that only one of the following two assertions holds true:
 - (a) If $|b(\varepsilon)| < b_0$ ($b(\varepsilon)$ as in (4.12)), there is a $\lambda_2 > \lambda_0^\varepsilon$ such that
 - (i) for $\lambda_0^\varepsilon < \lambda < \lambda_2$, the vector field $\tilde{v}_2^\lambda = (W + u_{2z}^\lambda, u_{2r}^\lambda)$ has the topological structure near $r = r_1$ (resp. $r = r_2$) as shown in Figure 5.2, and
 - (ii) for $\lambda_2 < \lambda$, there is a unique center of \tilde{v}_2^λ separated from each point $(\bar{z}_n, r_1) \in \partial M$ (resp. $(z_n, r_2) \in \partial M$) for $n = 0, 1, \dots, k_0$, as shown in Figure 5.3.
 - (b) If $|b(\varepsilon)| > b_0$, then for $\lambda > \lambda_2 = \lambda_0^\varepsilon$, the vector field \tilde{v}_2^λ is topologically equivalent to the structure as shown in Figure 5.4.

- (3) *There exists a $\lambda_3 > \lambda_1$ ($\lambda_1 \geq \lambda_2$) such that for $\lambda_i < \lambda < \lambda_3$, the vector field \tilde{v}_i^λ ($i = 1, 2$) is topologically equivalent to the structure as shown in Figure 5.4, and λ_3 is independent of γ .*

From Theorems 4.1, 5.5 and 5.3, we immediately derive the following theorem, which links the dynamics with the structure in the physical space.

Theorem 5.6. *Assume that the conditions in Theorem 4.1 hold true, $W_0 = 0$ in (2.7), and $0 < \gamma < \gamma_0$. Then there are $\tilde{\lambda}_j$ ($1 \leq j \leq 3$) with $\tilde{\lambda}_1 = \lambda_1$, $\lambda^* \leq \tilde{\lambda}_2 \leq \lambda_2$, $\tilde{\lambda}_3 = \lambda_3$, where λ_j ($1 \leq j \leq 3$) are as in Theorem 5.5 and λ^* is the saddle-node bifurcation point (if it exists), otherwise $\lambda^* = \lambda_0^\varepsilon$, and for each $\lambda < \tilde{\lambda}_3$ the space H can be decomposed into two open sets*

$$H = \overline{U}_1^\lambda + \overline{U}_2^\lambda, \quad U_1^\lambda \cap U_2^\lambda = \emptyset,$$

such that the following assertions hold true.

- (1) *For $\varphi \in U_i^\lambda$ there is a time $t_0 \geq 0$ such that if $\lambda < \tilde{\lambda}_i$ ($i = 1, 2$) for the solution $u(t, \varphi) = (\tilde{u}, u_\theta)$ of (2.2) with (2.7) and (2.23), the vector field \tilde{u} is topologically equivalent to the structure as shown in Figure 5.2, and if $\tilde{\lambda}_i < \lambda < \tilde{\lambda}_3$, \tilde{u} is topologically equivalent to the structure as shown in Figure 5.4 for all $t > t_0$.*
- (2) *In particular, for the initial value $\varphi \in U_i^\lambda$ near the Couette–Poiseuille flow (2.3), if $\lambda > \tilde{\lambda}_i$ there is a time $t_0 > 0$ such that the solution $u(t, \varphi)$ of (2.2) with (2.7) and (2.23) has a boundary-layer separation at $t = t_0$, i.e., the vector field \tilde{u} of $u(t, \varphi)$ is topologically equivalent to the structure as shown in Figure 5.2 for $t < t_0$, and \tilde{u} is topologically equivalent to the structure near boundary ∂M as shown in Figure 5.3 for $t_0 < t$.*

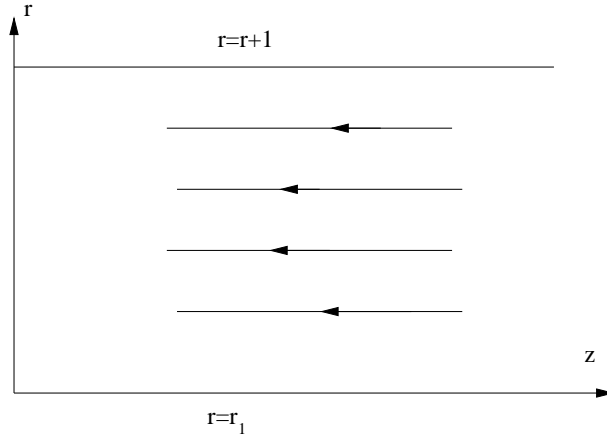


FIGURE 5.2. Vertical shear flow with vanishing boundary velocity

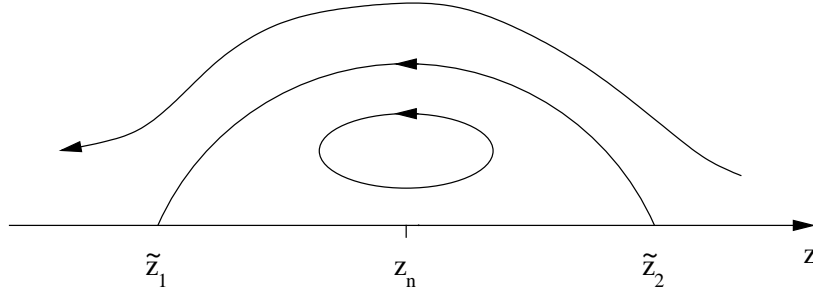


FIGURE 5.3. Boundary separation and re-attachment.

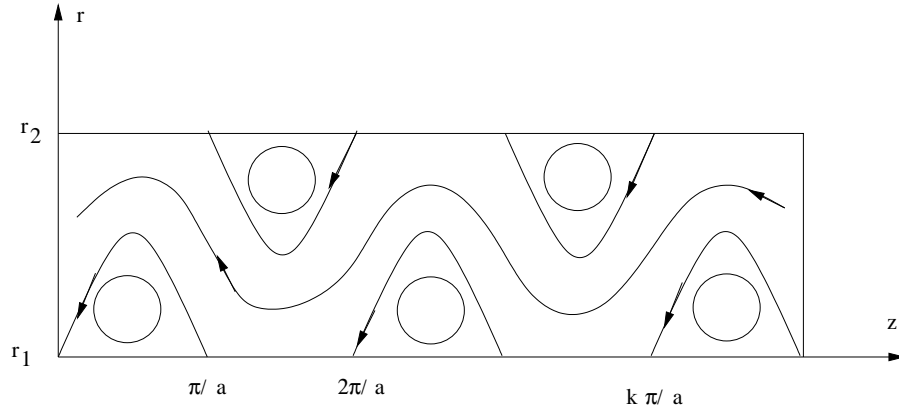


FIGURE 5.4. Propagating Taylor Vortices after boundary-layer separations.

Remark 5.7. The critical values $\tilde{\lambda}_i$ ($i = 1, 2$) in Theorem 5.6 which determine the boundary-layer separation for the Taylor-Couette-Poiseuille flow problem depend continuously on the non-dimensional parameter $\gamma = p_0 d^3 / 4\rho\nu^2$. The expansion is as follows

$$(5.6) \quad \tilde{\lambda}_i(\gamma) = \alpha_i \gamma + o(|\gamma|), \quad \alpha_i > 0 \quad \text{a constant, } i = 1, 2.$$

In addition, the condition $0 < \gamma < \gamma_0$ in the above theorems is the consequence of the narrow-gap assumption (i.e. d is small).

Remark 5.8. The height L has an integer number times the length of π/a , i.e., $L = K \cdot \frac{\pi}{a}$ for some integer K . Hence the number of vortices separated from $r = r_1$ and $r = r_2$ in Theorems 5.5 and 5.6 satisfies

$$k_0 = \begin{cases} \left\lceil \frac{K+1}{2} \right\rceil & \text{for } r = r_1 \text{ (or } r = r_2), \\ \left\lfloor \frac{K}{2} \right\rfloor & \text{for } r = r_2 \text{ (or } r = r_1). \end{cases}$$

where $[\alpha]$ is the integer part of n .

5.2.2. *Proof of Theorem 5.5.* We proceed in a few steps as follows.

STEP 1. By Theorem 4.1, as the Taylor number $T = \lambda^2$ satisfies $T_c < T$ ($T_c = \lambda_0^{\varepsilon^2}$), or equivalently $\lambda_0^\varepsilon < \lambda$, the equations (2.2) with (2.7) and (2.14) generate from the basic flow (2.3) two steady state solutions

$$v_i^\lambda = \left(W(r) + u_{iz}^\lambda, u_{ir}^\lambda, V(r) + u_{i\theta}^\lambda \right), \quad i = 1, 2,$$

which are asymptotically stable under the axisymmetric perturbation.

It is clear that the number $b(\varepsilon)$ in (4.12) satisfies

$$\lim_{\varepsilon \rightarrow 0} b(\varepsilon) = 0, \quad (\varepsilon = (1 - \mu, \gamma)).$$

Hence, to prove this theorem, it suffices to proceed only for the following two vector fields:

$$(5.7) \quad \tilde{V}_1^\lambda = (W(r), 0) + \frac{\sqrt{b^2 + 4\alpha(\lambda - \lambda_0^\varepsilon)} - b}{2C} \tilde{u}_0,$$

$$(5.8) \quad \tilde{V}_2^\lambda = (W(r), 0) + \frac{\sqrt{b^2 + 4\alpha(\lambda - \lambda_0^\varepsilon)} + b}{2C} \tilde{u}_0,$$

where $b = b(\varepsilon)$, $\alpha = \alpha(\varepsilon) > 0$ near $\varepsilon = 0$, $C > 0$ a constant,

$$\begin{aligned} W(r) &= -\gamma \bar{r}(1 - \bar{r}), \quad (\bar{r} = r - r_1), \\ \tilde{u}_0 &= \left(\frac{1}{a} \sin az R'(r), -\cos az R(r) \right), \end{aligned}$$

and $R(r)$ is given by (4.7).

STEP 2. *Claim:* \tilde{V}_1^λ has a boundary-layer separation on $\lambda_1 < \lambda$ for some $\lambda_1 > \lambda_0^\varepsilon$.

We shall apply the structural bifurcation theorem, Theorem 5.4, to prove this claim. To this end, let

$$\Lambda = \left(\sqrt{b^2 + 4\alpha(\lambda - \lambda_0^\varepsilon)} - b \right) / 2aC,$$

which is an increasing function of λ with $\Lambda = 0$ at $\lambda = \lambda_0^\varepsilon$. Thus the vector field (5.7) can be rewritten as

$$(5.9) \quad \tilde{V}_1^\lambda = (\Lambda \sin az R'(r) - \gamma \bar{r}(1 - \bar{r}), -a\Lambda \cos az R(r)).$$

We discuss the structural bifurcation of (5.9) only on the portion $r = r_1$ of ∂M , and on this portion $r = r_1 + 1$ of ∂M , the proof is the same.

Since $R(r_1) = R'(r_1) = 0$, the zero points of $\frac{\partial \tilde{V}_1^\lambda}{\partial n} = \frac{\partial \tilde{V}_1^\lambda}{\partial r}$ on $r = r_1$ are given by the solutions of the equation

$$(5.10) \quad \Lambda \sin az R''(r_1) - \gamma = 0.$$

Let $\Lambda_0 = \gamma/R''(r_1)$. Then, we infer from (5.10) that for $r = r_1$,

$$(5.11) \quad \frac{\partial \tilde{V}_1^\lambda}{\partial n} \begin{cases} \neq 0 & \text{when } \Lambda < \Lambda_0, \\ = 0 & \text{when } \Lambda = \Lambda_0 \text{ and } z_n = \frac{(4n+1)\pi}{2a}. \end{cases}$$

As in (5.1), \tilde{V}_1^λ can be Taylor expanded at $\Lambda = \Lambda_0$ as follows:

$$(5.12) \quad \begin{cases} \tilde{V}_1^\lambda = V^0 + (\Lambda - \Lambda_0)V^1, \\ V^0 = (V_z^0, V_r^0) = (\Lambda_0 \sin azR' - \gamma\bar{r}(1 - \bar{r}), -a\Lambda_0 \cos azR), \\ V^1 = (V_z^1, V_r^1) = (\sin azR', -a \cos azR). \end{cases}$$

We derive from (4.7) that

$$(5.13) \quad \begin{aligned} R''(r) = & -\alpha_0^2 \cos \alpha_0 x \\ & + [\beta_1(\alpha_2^2 - \alpha_1^2) + 2\beta_2\alpha_1\alpha_2] \cosh(\alpha_1 x) \cos(\alpha_2 x) \\ & + [\beta_2(\alpha_1^2 - \alpha_2^2) + 2\beta_1\alpha_1\alpha_2] \sinh(\alpha_1 x) \sin(\alpha_2 x), \end{aligned}$$

where $x = \bar{r} - \frac{1}{2}$. Inserting (4.8) into (5.13) we obtain

$$(5.14) \quad R''(r_1) \simeq 8 + 1.7e^{\frac{5}{2}} \simeq 28.$$

Thus we derive from (5.11), (5.12) and (5.14) that

$$\begin{aligned} \frac{\partial V^0}{\partial n} = \frac{\partial V^0}{\partial r} = 0 & \quad \text{at } (z_n, r_1) = \left(\frac{4n+1}{2a} \pi, r_1 \right), \\ \frac{\partial V^1}{\partial n} = \frac{\partial V^1}{\partial r} \simeq (28, 0) & \quad \text{at } (z_n, r_1), \\ \frac{\partial^3 V_z^0}{\partial \tau^2 \partial n} = \frac{\partial^3 V_z^0}{\partial z^2 \partial r} \simeq -28a^2 \Lambda_0 & \quad \text{at } (z_n, r_1). \end{aligned}$$

Namely, the conditions (5.2), (5.4) and (5.5) are valid.

Now, we verify the condition (5.3), i.e., we need to prove that

$$(5.15) \quad \text{ind} \left(\frac{\partial \tilde{V}_1^{\lambda_0}}{\partial r}, (z_n, r_1) \right) = 0.$$

To this end, we consider the equation

$$(5.16) \quad \frac{\partial V_z^\lambda}{\partial r} = \Lambda \sin azR''(r) - \gamma(1 - 2\bar{r}) = 0.$$

The function $R''(r)$ has the Taylor expansion at $r = r_1$ as follows

$$(5.17) \quad R''(r) = R''(r_1) + R''(r_1)\bar{r} + o(|\bar{r}|).$$

From (4.7) and (4.3) we can derive that

$$(5.18) \quad R'''(r_1) \simeq -64 \times \frac{1.7}{2} - 10 \times e^{\frac{5}{2}} \simeq -175.$$

On the other hand, we note that

$$\Lambda = \frac{\gamma\Lambda}{R''(r_1)\Lambda_0} = \frac{\gamma\Lambda}{28\Lambda_0}.$$

Thus, by (5.14), (5.17) and (5.18), the equation (5.16) near $(z, r) = (z_n, r_1)$ can be rewritten in the following form

$$(5.19) \quad \left(1 - \frac{\Lambda}{\Lambda_0} \sin az\right) + \left(\frac{175\Lambda \sin az}{28\Lambda_0} - z\right) \bar{r} + o(|\bar{r}|) = 0.$$

Obviously, there is a $\delta > 0$ such that the equation (5.19) has no solutions in $0 < r - r_1 = \bar{r} < \delta$, $|z - z_n| < \delta$ and $\Lambda < \Lambda_0$. Namely, in a neighborhood of (z_n, r_1) , the vector field $\frac{\partial \tilde{V}_1^\lambda}{\partial r}$ has no singular points for $\Lambda < \Lambda_0$. Hence, by the invariance of the index sum of singular points in a domain, one can get (5.15).

Let λ_1 be the number that $\Lambda(\lambda_1) = \Lambda_0$. It is clear that $\lambda_1 > \lambda_0^\varepsilon$. Thus, this claim follows from Theorem 5.4.

STEP 2. The vector field \tilde{V}_1^λ has no singular points in M for $\lambda_0^\varepsilon < \lambda < \lambda_1$, and has only one singular point in each domain $(z_k, z_{k+1}) \times (r_1, r_1 + 1)$, $z_k = \frac{k\pi}{a}$ ($0 \leq k \leq K = \frac{aL}{\pi} - 1$), for all $\lambda > \lambda_1$.

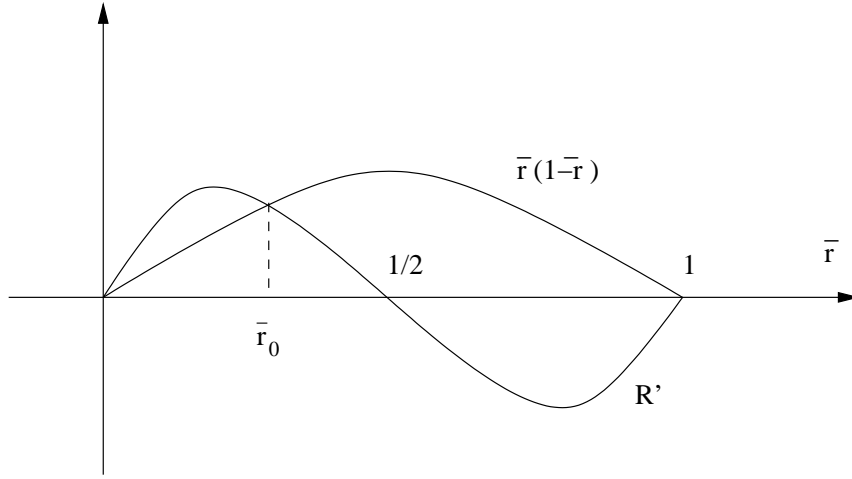
From (5.9) we find that the singular points of \tilde{V}_1^λ must be in the lines $\ell_k = \left\{ \left(\frac{k\pi}{a} + \frac{\pi}{2a}, r \right) \mid r_1 < r < r_1 + 1 \right\} \subset M$, and

$$\tilde{V}_{1r}^\lambda = 0, \quad \tilde{V}_{1k}^\lambda = (-1)^k \Lambda R'(r) - \gamma \bar{r}(1 - \bar{r}), \quad \text{on } \ell_k.$$

Therefore, to prove this claim we only consider this equation:

$$(5.20) \quad (-1)^k \frac{R'(r)}{R''(r_1)} - \frac{\Lambda_0}{\Lambda} \bar{r}(1 - \bar{r}) = 0, \quad 0 < \bar{r} < 1.$$

It is known that $R'(r)$ is odd and $\bar{r}(1 - \bar{r})$ is an even function on the variable $x = \bar{r} - \frac{1}{2}$. These functions are illustrated by Figure 5.5.

FIGURE 5.5. \bar{r}_0 is a zero point of (5.20).

Direct computations show that (5.20) has no zero points in $0 < \bar{r} < 1$ for $\Lambda \leq \Lambda_0$, and there is a unique zero point $\bar{r}_0 \in (0, 1)$ for all $\Lambda_0 < \Lambda$. Furthermore, $0 < \bar{r}_0 < \frac{1}{2}$ as $k = \text{even}$, and $\frac{1}{2} < \bar{r}_0 < 1$ as $k = \text{odd}$. Thus, this claim is proved.

STEP 3. By the invariance of index sum of singular points in a domain, it is easy to see that the singular point of \tilde{V}_1^λ in the domain $(z_k, z_{k+1}) \times (r_1, r_2)$ must be a center, which is enclosed by an orbit of \tilde{V}_1^λ connected to both boundary saddle points \tilde{z}_1 and \tilde{z}_2 , as shown in Figure 5.3, \tilde{z}_1, \tilde{z}_2 are on $r = r_1$ as $k = \text{even}$, and on $r = r_2$ as $k = \text{odd}$.

Finally, by Steps 1-3 one readily derives this theorem. The proof is complete.

6. STRUCTURAL TRANSITION OF THE COUETTE-POISEUILLE FLOW: INTERIOR SEPARATIONS

In this section, we study the structure and its transitions in the TCP flow for the case where the z -direction boundary velocity $W_0 \neq 0$ in (2.7). We show that the secondary flow from the Couette–Poiseuille flow (2.3) will have interior separations as the Taylor number T exceeds some critical value \tilde{T} which is an increasing function of the nondimensional parameter $\gamma = p_0 d^3 / 4\rho\nu$. For this purpose, we recall some rigorous analysis by the authors on interior separations of incompressible flows; see [10, 12].

6.1. Interior structural bifurcation of 2D incompressible Flows.

Following notations used in Section 5.1, we let $u(\cdot, \lambda) \in D^r(M, \mathbb{R}^2)$ ($r \geq 1$) have the Taylor expansion at $\lambda = \lambda_0$ as follows

$$(6.1) \quad u(x, \lambda) = u^0(x) + (\lambda - \lambda_0)u^1(x) + o(|\lambda - \lambda_0|).$$

We assume that $x_0 \in M$ is an isolated singular point of $u_0(x)$, and

$$(6.2) \quad \text{ind}(u^0, x_0) = 0,$$

$$(6.3) \quad Du^0(x_0) = \begin{pmatrix} \frac{\partial u_1^0(x_0)}{\partial x_1} & \frac{\partial u_1^0(x_0)}{\partial x_2} \\ \frac{\partial u_2^0(x_0)}{\partial x_1} & \frac{\partial u_2^0(x_0)}{\partial x_2} \end{pmatrix} \neq 0,$$

$$(6.4) \quad u^1(x_0) \cdot e_2 \neq 0,$$

where e_2 is a unit vector satisfying

$$(6.5) \quad \begin{cases} Du^0(x_0)e_2 = \alpha e_1 & (\alpha \neq 0), \\ Du^0(x_0)e_1 = 0. \end{cases}$$

We also assume that $u^0 \in C^m$ at $x_0 \in M$ for some even number $m \geq 2$, and

$$(6.6) \quad \frac{\partial^k(u^0(x_0) \cdot e_2)}{\partial e_1^k} = \begin{cases} 0, & 1 \leq k < m = \text{even}, \\ \neq 0, & k = m = \text{even}. \end{cases}$$

Then we have the following interior structural bifurcation theorem, which was proved in [10, 12].

Theorem 6.1. *Let $u(\cdot, \lambda) \in D^r(M, \mathbb{R}^2)$ be as given in (6.1) satisfying the conditions (6.2)-(6.4). Then the following assertions hold true.*

- (1) *As $\lambda < \lambda_0$ (or $\lambda > \lambda_0$), the flow described by $u(x, \lambda)$ is topologically equivalent to a tubular flow near $x_0 \in M$ as shown in Figure 6.1(a).*
- (2) *As $\lambda > \lambda_0$ (or $\lambda < \lambda_0$) there must be some centers of $u(x, \lambda)$ separated from $x_0 \in M$ as shown schematically in either Figure 6.1 (c) or (d).*
- (3) *The centers are enclosed by an extended orbit $\gamma(\lambda)$, and $\gamma(\lambda)$ shrinks to x_0 as $\lambda \rightarrow \lambda_0$.*
- (4) *If the condition (6.6) is satisfied, then the center separated from $x_0 \in M$ is unique, as shown in Figure 6.1 (c).*

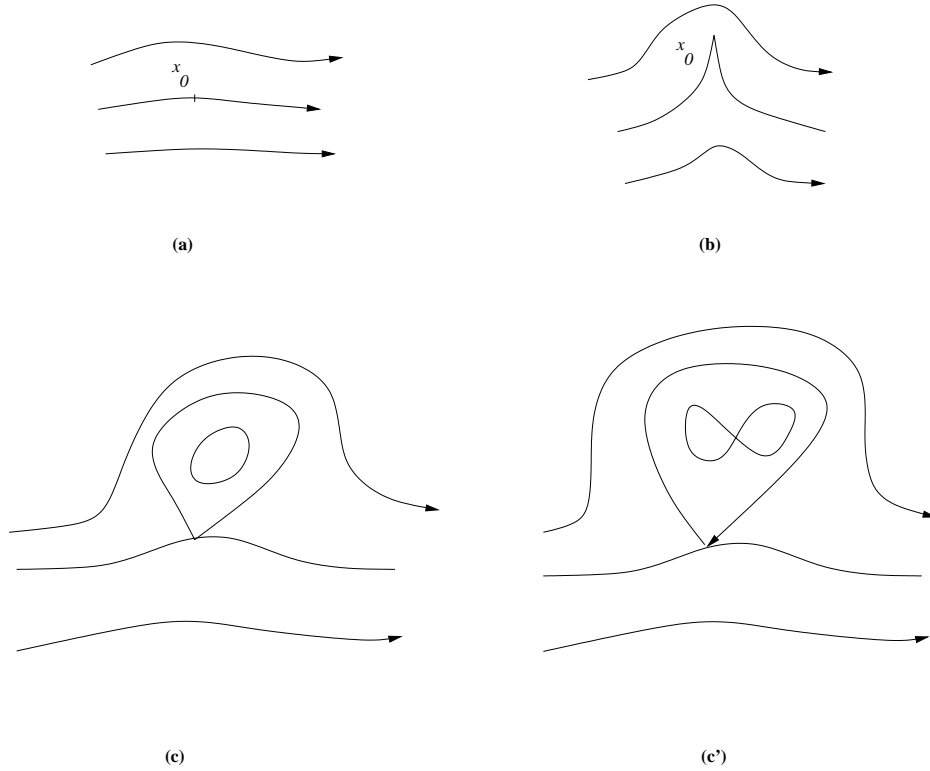


FIGURE 6.1. Interior separation.

6.2. Interior separation of the Taylor-Couette-Poiseuille Flow.

6.2.1. Main theorems. Let the constant velocity $W_0 \neq 0$ in the boundary condition (2.7). Then, we have the following interior separation theorems for the TCP flow problem. The results obtained here are in agreement with the numerical results obtained by Raguin and Georgiadis; see e.g. Figure 8 in [13].

Theorem 6.2. Assume that the conditions in Theorem 4.1 hold true, and let $W_0 \neq 0$ in (2.7). Then there exists $\gamma_0 > 0$ such that if $0 < \gamma < \gamma_0$, then for the two (bifurcated) steady state solutions $v_1^\lambda = \mathcal{U}^{cp} + u_i^\lambda$ ($i = 1, 2$) of (2.2) and (2.7), the following assertions hold true:

- (1) For v_1^λ , there is a $\lambda_1 > \lambda_0^\varepsilon$ such that
 - (a) for $\lambda_0^\varepsilon < \lambda < \lambda_1$, the vector field $\tilde{v}_1^\lambda = (W + u_{1z}^\lambda, u_{1r}^\lambda)$ is topologically equivalent to the structure as shown in Figure 6.2, and
 - (b) for $\lambda_1 < \lambda$, there is exactly a pair of center and saddle points separated from a point in each domain $(\tilde{z}_k, \tilde{z}_{k+1}) \times (r_1, r_2) \subset M$, with $\tilde{z}_k = \frac{k\pi}{a}$ ($0 \leq k \leq K$), as shown in Figure 6.1(c). Moreover, the value λ_1 is a continuous and increasing function of γ .

- (2) For v_2^λ there is a number $b_0 > 0$ such that only one of the following two assertions holds true.
- (a) If $|b(\varepsilon)| < b_0$, where $b(\varepsilon)$ is given by (4.12), then there is a $\lambda_2 > \lambda_0^\varepsilon$ ($\lambda_1 \geq \lambda_2$) such that the same conclusions as Assertion (1) holds true for v_2^λ with λ_2 replacing λ_1 .
 - (b) If $|b(\varepsilon)| > b_0$, then when $\lambda > \lambda_2 = \lambda_0^\varepsilon$, the vector field $\tilde{v}_2^\lambda = (W + u_{2z}^\lambda, u_{2r}^\lambda)$ is topologically equivalent to the structure as shown in Figure 6.3.
- (3) There exists a $\lambda_3 > \lambda_1$, with λ_3 independent of γ , such that if $\lambda_i < \lambda < \lambda_3$, the vector field \tilde{v}_i^λ ($i = 1, 2$) is topologically equivalent to the structure as shown in Figure 6.3.

The following theorem is a direct corollary of Theorems 5.3, 4.1 and 6.2, and provides a link between the dynamics and the structure in the physical space.

Theorem 6.3. *Assume that the conditions in Theorem 4.1 hold true, $W_0 \neq 0$ in (2.7), and $0 < \gamma < \gamma_0$. Then there are $\tilde{\lambda}_j$ ($1 \leq j \leq 3$) with $\tilde{\lambda}_1 = \lambda_1$, $\lambda^* \leq \tilde{\lambda}_2 \leq \lambda_2$, $\tilde{\lambda}_3 = \lambda_3$, and for each $\lambda < \tilde{\lambda}_3$ the space H can be decomposed into two open sets*

$$H = \overline{U}_1^\lambda + \overline{U}_2^\lambda, \quad U_1^\lambda \cap U_2^\lambda = \emptyset,$$

such that the following assertions hold true.

- (1) For $\varphi \in U_i^\lambda$ there is a time $t_0 \geq 0$ such that when $t > t_0$, for the solution $u(t, \varphi) = (\tilde{u}, u_\theta)$ of (2.2) with (2.7) and (2.23), as $\lambda < \tilde{\lambda}_i$ the vector field \tilde{u} is topologically equivalent to the structure as shown in Figure 6.2, and as $\tilde{\lambda}_i < \lambda < \tilde{\lambda}_3$ ($i = 1, 2$) \tilde{u} is topologically equivalent to the structure as shown in Figure 6.3.
- (2) In particular, for the initial value $\varphi \in U_i^\lambda$ near the Couette–Poiseuille flow (2.3), and for $\lambda > \tilde{\lambda}_i$ ($i = 1, 2$), there is a time $t_0 > 0$ such that for the solution $u(t, \varphi) = (\tilde{u}, u_\theta)$ of (2.2) with (2.7) and (2.23), \tilde{u} has an interior separation at $t = t_0$ from a point in each domain $(\tilde{z}_k, \tilde{z}_{k+1}) \times (r_1, r_2) \subset M$, where $\tilde{z}_k = \frac{k\pi}{a}$, as described by Theorem 6.2 with t replacing λ .

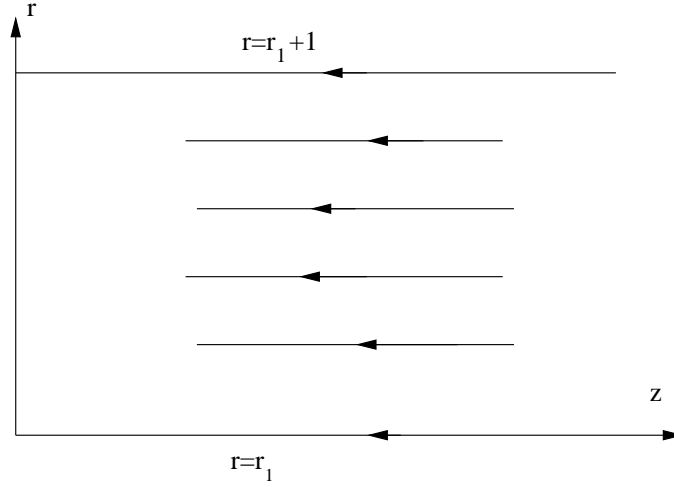


FIGURE 6.2. Vertical shear flow with a constant velocity $W_0 \neq 0$ on boundary.

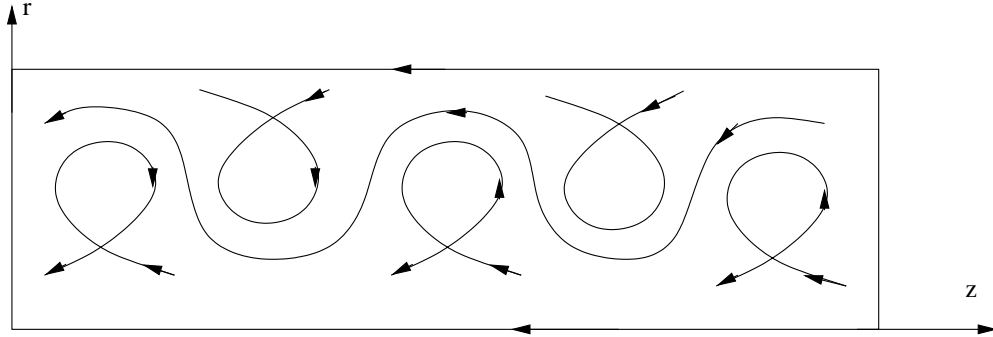


FIGURE 6.3. Propagating Taylor vortices separated from the interior.

6.2.2. *Proof of Theorem 6.2.* This proof is similar to that of Theorem 5.5. Here, we only need to prove the interior structural bifurcation in Assertion (1) for the following vector field

$$(6.7) \quad \tilde{V}_1^\lambda = (\Lambda \sin az R'(r) - \gamma \bar{r}(1 - \bar{r}) - \gamma W_0, -a\Lambda \cos az R(r)).$$

We proceed by applying Theorem 6.1. Consider the equations

$$(6.8) \quad \begin{cases} \pm \Lambda R'(r) - \gamma \bar{r}(1 - \bar{r}) - \gamma W_0 = 0, \\ \pm \Lambda R''(r) - \gamma(1 - 2\bar{r}) = 0. \end{cases}$$

By computing, we can know that the equations (6.8) have a unique solution (Λ_0, r_0^\pm) with $r_0^+ - r_1 = 1 - (r_0^- - r_1)$ and $r_0^+ < r_*^+ < r_*^- < r_0^-$, where r_*^\pm are the maximum points of $R'(r)$, i.e.

$$(6.9) \quad R''(r_*^\pm) = 0, \quad (r_*^+ \simeq r_1 + \frac{1}{5}, r_*^- \simeq r_1 + \frac{4}{5}).$$

The vector field (6.7) has the following expression at $\Lambda = \Lambda_0$

$$(6.10) \quad \begin{cases} \tilde{V}_1^\lambda = V^0 + (\Lambda - \Lambda_0)V^1, \\ V^0 = (V_z^0, V_r^0) = (\Lambda_0 \sin azR' - \gamma\bar{r}(1 - \bar{r}) - \gamma W_0, -a\Lambda_0 \cos azR), \\ V^1 = (V_z^1, V_r^1) = (\sin azR', -a \cos azR). \end{cases}$$

It is easy to see that for the solution (Λ_0, r_0^\pm) of (6.8) the points (z_{2k}, r_0^+) and $(z_{2k+1}, r_0^-) \in M$ are singular points of V^0 , where $z_m = \frac{m\pi}{a} + \frac{\pi}{2a}$.

For simplicity, we only consider the structural bifurcation of V^0 at the singular point $(z_0, r_0) = (\frac{\pi}{2a}, r_0^+)$.

We can check that when $\Lambda < \Lambda_0$, the vector field \tilde{V}_1^λ given by (6.7) has no singular point in M , and (z_0, r_0) is an isolated singular point of \tilde{V}_1^λ at $\Lambda = \Lambda_0$. Therefore we have

$$(6.11) \quad \text{ind}(V^0, (z_0, r_0)) = 0.$$

In addition, we see that

$$(6.12) \quad DV^0(z_0, r_0) = \begin{pmatrix} \frac{\partial V_z^0}{\partial z} & \frac{\partial V_z^0}{\partial r} \\ \frac{\partial V_r^0}{\partial z} & \frac{\partial V_r^0}{\partial r} \end{pmatrix}_{(z_0, r_0)} = \begin{pmatrix} 0 & 0 \\ a^2 \Lambda R(r_0) & 0 \end{pmatrix} \neq 0$$

and the eigenvectors e_1 and e_2 satisfying

$$\begin{cases} DV^0(z_0, r_0)e_2 = \alpha e_1, & (\alpha = a^2 \Lambda R(r_0)), \\ DV^0(z_0, r_0)e_1 = 0, \end{cases}$$

are given by

$$e_1 = (0, 1), \quad e_2 = (1, 0).$$

Hence, we have

$$(6.13) \quad V^1(z_0, r_0) \cdot e_2 = V_z^1(z_0, r_0) = R'(r_0) \neq 0.$$

From (6.11)–(6.13) we find that the conditions (6.2)–(6.4) are satisfied by the vector field \tilde{V}_1^λ at $\Lambda = \Lambda_0$ and $(z, r) = (z_0, r_0)$.

Finally, we verify the condition (6.6). By (6.8) we have

$$(6.14) \quad \left. \frac{\partial^k (V^0 \cdot e_2)}{\partial e_1^k} \right|_{(z_0, r_0)} = \frac{\partial^k V_z^0(z_0, r_0)}{\partial r^k} = 0 \quad \text{for } k = 0, 1,$$

and

$$(6.15) \quad \frac{\partial^2 V_z^0(z_0, r_0)}{\partial r^2} = \Lambda_0 R'''(r_0) + 2\gamma.$$

By (6.8) we see that

$$(6.16) \quad \frac{\gamma}{\Lambda_0} = \frac{R''(r)}{1 - 2(r_0 - r_1)}.$$

On the other hand, we can check that

$$(6.17) \quad -\frac{R'''(r)}{R''(r)} > \frac{2}{1-2(r-r_1)}, \quad \forall r_1 \leq r < r_*^+,$$

where r_*^+ satisfies (6.9). From (6.15)–(6.17) it follows that

$$(6.18) \quad \frac{\partial^2 V_z^0(z_0, r_0)}{\partial r^2} \neq 0.$$

Thus, from (6.14) and (6.18) we derive the condition (6.6) with $m = 2$. By Theorem 6.1, the vector field \tilde{V}_1^λ has an interior separation from each point (y_m, r_0) at $\Lambda = \Lambda_0$, where $r_0 = r_0^+$ if $y_m = \frac{2k\pi}{a} + \frac{\pi}{2a}$ and $r_0 = r_0^-$ if $y_m = \frac{(2k+1)\pi}{a}$. The proof is complete. \square

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